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Math 6230: Differential Games, Optimal Control and Front Propagation Professor Vladmirsky

Question 27

1. Assuming that Ω is convex, prove that for any x1, x2 in Ω\d c

The PDE satisfies:

with the following bounds on K, F and q: 0<F1< f(<F2 , 0<K1<K<K2 and 0<Q1 < Q<Q2.

Divide each side by to get:

Now, consider moving from point , to along the unit vector u where u points in the direction x1 to x2.

The directional derivative is given by the following

Thereby the infinitesimal slope between and is bounded by and . Therefore, the absolute average slope between any two points x1 and x2 along the unit vector from x1 to x2 is bounded by the ratios and .

Effectively, the bound above says that if two points are close together their exit times must be sufficiently close together. To break the condition we merely need to pick boundary points such that though close together have drastically different values. Consider the domain Ω where there are two points on dΩ x1 and x2 with distance r1 given and , with q(x1) = 0. We can then set or we can set and the bound would not hold implying that there is no solution by a contradiction. We must then require that for every pair of points on the boundary x1 and x2 the bound above holds.

Xy

Xx1

Xx2

Xr

Xr2

Xr1

Q(x) must be Lipschitz continuous with the scaling factor higher K= and lower K=. You can also prove a similar property if Ω is not convex. In this case instead of taking the shortest distance in a convex set which is a line with length we can bound it by taking the length of the shortest path from x1 to x2 . The above proof holds in that you move along the shortest path and two neighboring points can only vary by the gradient bound provided above.

(B) Let d(x) be the distance from x to dΩ. Show that if Then

We are given that

The bound above states that the minimum time exit is bounded from below by the time needed to travel the shortest distance from x to the boundary plus the minimum boundary exit time and bounded above by the time needed to travel from x to the boundary along with the slowest speed path plus the maximum exit time penalty.

Because the shortest path is a straight line from x to the boundary

Now for the maximum bound. For any path it is trivially clear that if we were to set and and q(\*,\*)=q2 the new v(x) would be greater than the true v(x). Therefore

can then be chosen to minimize the value of the integral. This evaluates to d(x), yielding:

q2

(C) If y(t) is an optimal trajectory then v(y(t)) is a strictly decreasing function.

To show v(y(t)) is a strictly decreasing function of t, take the derivative with respect to t.

The optimal trajectory means the following. Let the starting point of the trajectory be at y0. y(t) is the solution to

where is the a that minimizes the cost function

This can be looked at in the following way. For every level set of minimum exit time, a(\*) guides the trajectory in minimum time to the next level set. We can then iterate the problem using the intersection of the minimum time trajectory of the inner problem as the starting point of the outer problem. We set the v(x)=q(x)=Ci which is the ith level set value.

Y0

specified on boundary

Path of Minimum Time

Level Sets of Equal Exit Time

Back to the above equations, substituting

The PDE gives us:

Suppose that

for some time t0 implying that v(y(t)) is increasing at time t0, then:

since the PDE would not be satisfied when the left hand term evaluates to non-negative number when

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However, this is a contradiction since this implies that is not the optimal trajectory. With the geometric formulism above, this means that v(y(t)) is not approaching a new lower exit time level set, since there exists that takes (t) less time to get to the next level set. Then select this new 0<t<t0, and from t0<t<T use new and the from t0<t<T(y0,) . Therefore it is clear by contradiction that

(D)Given any constant C such that the trajectory y(t) will intersect the level set v(x)=C at some point and if is distance d1 away from that set, show:

x1

Let x1 denote the closest point to on the level set v(x)=C. We have that

by definition of being a level set. We know that v(y(t)) varies continuously from v( to q where q is below We are guaranteed v(y(t))=C for some t.

From above it is known that the optimal trajectory in the problem for the entire domain is the same as the optimal trajectory from to the level set where the new problem v(x)=q\_new(x)=C. We can then apply the theorem in part (B).

= and therefore we get: